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A new analytical solution for heat transfer in the entrance region of ducts: hydrodynamically developed flows of power-law fluids with constant wall temperature

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Abstract—Heat transfer by forced convection in the thermal entrance of flat ducts and circular pipes is investigated for constant surface temperatures and hydrodynamically developed flows. A new technique, based on separation of variables and spectral decomposition of the eigenfunction in polynomial form, is introduced to solve the problem for viscous fluids. Application of the present method is discussed for Newtonian and power-law fluids. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

Forced convection heat transfer in the entrance region of ducts has received considerable attention due to its practical importance, and numerous analytical as well as numerical studies were devoted to the Graetz problem. Sellars *et al.* [1] discussed an asymptotic method which gives eigenvalues of high-order for fully-developed laminar flow of Newtonian fluids in circular tube or parallel-plate channel under constant wall temperature boundary condition. Blackwell [2] numerically solved the Sturm–Liouville system for a Bingham fluid, and more recently Johnston [3] examined, anew, the same problem analytically by the use of separation of variables and took into account axial conduction. However, this method requires considerable numerical work to be completed and its relevance is questionable, since the numerical solution presented by Bilir [4] for the Graetz problem showed that axial conduction is important only very close to the inlet section. Ramachandran [5] used a method based on contour integrals in conjunction with a step by step, numerical procedure along the duct axis. Nguyen [6] presented results of a numerical study of simultaneously developing flow in a circular tube, accounting for axial diffusion of momentum and heat.

In the present study, we consider ducts in which the dimensionless, developed velocity profile scaled by u_{max} may be written as:

$$U(Y) = \sum_{k=0}^{p} c_k Y^k. \tag{1}$$

Obviously, for Newtonian and power-law fluids $U(Y)$ is given by $U(Y) = 1 - Y^{\alpha}$ where $\alpha = 2$ for a Newtonian fluid and $\alpha = (1+n_c)/n_c$ for power-law fluids. For non-integer values of α, $U(Y)$ could be expressed in the polynomial form as shown in equation (1) through the use of a Lagrangian interpolation procedure or with an approximation procedure (Weierstrass approximation theorem) if the derivative of order $(p+1)$ of $U(Y)$ is not continuous in the interval $[0, 1]$.

For the dynamically developed regime with the assumption of negligible axial conduction, the energy equation reduces to:

$$U(Y)\frac{\partial \theta}{\partial X} = \frac{1}{Y^m}\frac{\partial}{\partial Y}\left(Y^m \frac{\partial \theta}{\partial Y}\right) \tag{2a}$$

$$\theta(X, 1) = 0, \quad \left.\frac{\partial \theta}{\partial Y}\right|_{Y=0} = 0 \quad \text{and} \quad \theta(0, Y) = 1. \tag{2b}$$

In equation (2), the dimensionless axial coordinate is related to the Graetz number as $X = 32\ Gz^{-1}/3$ for parallel-plate channels ($m = 0$) and as $X = 2\ Gz^{-1}$ for circular pipes ($m = 1$).

NOMENCLATURE

a	fluid thermal diffusivity
D_h	hydraulic diameter
Gz	Graetz number, $Gz^{-1} = (x/D_h)/Pe$
L	half-width of flat duct
m	metric parameter ($m = 0$ for a flat duct, $m = 1$ for a circular pipe)
n_c	power-law index
Nu	local Nusselt number
Pe	Péclet number, $Pe = u_m D_h/a$
R	radius
T	temperature
T_e	inlet temperature
T_w	wall temperature
u	axial fluid velocity
U	dimensionless axial velocity, $U = u/u_{max}$
u_m	average fluid velocity
u_{max}	maximum velocity
x	axial coordinate
y	normal coordinate, from the centerline and towards the wall
X	dimensionless axial coordinate, $X = ax/u_{max}I^2$ ($I = L$ or R)
Y	dimensionless normal coordinate, $Y = y/I$ ($I = L$ or R).

Greek symbols

α	velocity parameter, $\alpha = 1+1/n_c$
μ_i	eigenvalue of order i
ψ_i	eigenfunction of order i
θ	dimensionless temperature, $\theta = (T-T_w)/(T_e-T_w)$
$\bar{\theta}$	dimensionless bulk temperature.

2. SOLUTION PROCEDURE

By using separation of variables, solution of equation (2) can be written as:

$$\theta(X, Y) = \sum_{i=0}^{\infty} A_i \exp(-\mu_i^2 X)\psi_i(Y). \qquad (3)$$

The eigenfunctions $\psi_i(Y)$ are solutions of the following eigenvalue system:

$$\frac{1}{Y^m}\frac{d}{dY}\left(Y^m \frac{d\psi_i}{dY}\right)+\mu_i^2 U(Y)\psi_i = 0 \qquad (4a)$$

$$\psi_i'(0) = 0 \quad \text{and} \quad \psi_i(1) = 0. \qquad (4b)$$

In the limit of $\alpha \to \infty$, $U = 1$ (slug flow) and the eigenfunctions are $\sin(\mu_i Y)$ and $\cos(\mu_i Y)$ for $m = 0$ and J_0 $(\mu_i Y)$, Y_0 $(\mu_i Y)$ for $m = 1$. In this straightforward case, the full solution is not shown here, and the reader is referred to Kakaç and Yener [7]. In the following sections, solutions which can be expressed in terms of these functions will be denoted as A-solutions.

In the present study, we are looking for eigenfunctions which could be expressed in series expansion as:

$$\psi_i = \sum_{n=0}^{\infty} a_n^{(i)} Y^n. \qquad (5)$$

Solutions based on equation (5) will be denoted B-solutions.

Such a decomposition of the eigenfunctions on a polynomial basis of $\mathbb{R}$ is quite easy to obtain. Obviously, in the limit of $\alpha \to \infty$, it is just a Taylor series expansion of the above mentioned eigenfunctions (A-solution).

In order to satisfy the eigenvalue system, coefficients $a_n^{(i)}$ must satisfy the recurrence relation:

$$a_0^{(i)} = 1 \qquad (6a)$$

$$a_1^{(i)} = 0 \qquad (6b)$$

$$(n+m+1)(n+2)a_{n+2}^{(i)} + \mu_i^2 \sum_{k=0}^{n} c_k a_{n-k}^{(i)} = 0 \qquad (6c)$$

with $c_k = 0$ for $k > p$. In addition, since $\psi_i(1) = 0$, the coefficients of the series expansion for ψ_i should satisfy the following condition:

$$\sum_{n=0}^{\infty} a_n^{(i)} = 0. \qquad (7)$$

The μ_i are solutions of this eigenvalue equation which is in polynomial form.

The numerical Newton method combined with the secant method was used to find the eigenvalues. Since the eigenvalue equation depends only on μ_i^2 in polynomial form, the maximum number of positive real eigenvalues can be known for each series expansion. The lowest truncation order gives an order of magnitude of the first eigenvalue μ_0. For example, the approximate value $\mu_0^2 = 4(m+3)(m+1)/(m+5)$ is found for a Newtonian fluid. Therefore, the finding of the μ_i is easier. Several orders of truncation were tested by trial and error so that the results were independent of the truncation order. The computations were carried out in double machine accuracy. Hence, for all cases discussed in the present paper, the $\psi_i(1)$-residue (equation (7)) was less than 10^{-10}.

It can be shown that the eigenfunctions $\psi_i(Y)$ form a complete orthogonal set with respect to the weight function $Y^m U(Y)$ over the interval [0, 1]. Therefore, the orthogonality relation yields

$$A_i = \frac{\int_0^1 Y^m U(Y)\psi_i \, dY}{\int_0^1 Y^m U(Y)\psi_i^2 \, dY}$$

$$= \frac{\sum_{k=0}^{p}\sum_{n=0}^{\infty}\frac{a_n^{(i)}c_k}{(k+n+m+1)}}{\sum_{k=0}^{p}\sum_{n1=0}^{\infty}\sum_{n2=0}^{\infty}\frac{a_{n1}^{(i)}a_{n2}^{(i)}c_k}{(k+n1+n2+m+1)}}. \quad (8)$$

3. HEAT TRANSFER

The local Nusselt number based on the hydraulic diameter is given by:

$$Nu(X) = \frac{-C}{\theta(X)}\frac{\partial\theta}{\partial Y}\bigg|_{Y=1} \quad (9)$$

with $C = 4$ for $m = 0$ and $C = 2$ for $m = 1$. The bulk temperature at any X-distance reads

$$\bar{\theta}(X) = \frac{\int_0^1 Y^m U(Y)\theta(X, Y)\, dY}{\int_0^1 Y^m U(Y)\, dY}. \quad (10)$$

Substituting equation (5) into equations (9) and (10) and performing the algebra yields:

$$\bar{\theta}(X) = \frac{\sum_{i=0}^{\infty} A_i e^{-\mu_i^2 x}\sum_{k=0}^{p}\sum_{n=0}^{\infty}\frac{a_n^{(i)}c_k}{(k+n+m+1)}}{\sum_{k=0}^{p}\frac{c_k}{(k+m+1)}} \quad (11)$$

and

$$Nu(X) = \frac{-C}{\bar{\theta}(X)}\sum_{i=0}^{\infty} A_i e^{-\mu_i^2 x}\sum_{n=1}^{\infty} n a_n^{(i)}. \quad (12)$$

For fully-developed regime ($X \to \infty$), the asymptotic value of the Nusselt number becomes:

$$Nu_\infty = -C\frac{\sum_{n=1}^{\infty} n a_n^{(0)}\sum_{k=0}^{p}\frac{c_k}{(k+m+1)}}{\sum_{k=0}^{p}\sum_{n=0}^{\infty}\frac{a_n^{(0)}c_k}{(k+n+m+1)}} \quad (13)$$

and for a slug flow ($U = 1$ or $p = 0$) Nu_∞ reduces to:

$$Nu_\infty = -\frac{C}{m+1}\frac{\sum_{n=1}^{\infty} n a_n^{(0)}}{\sum_{n=0}^{\infty}\frac{a_n^{(0)}}{(n+m+1)}}. \quad (14)$$

4. RESULTS

The asymptotic Nusselt numbers are readily obtained by using the A-solution for slug flow in parallel-plate channel or circular pipe. These values are $Nu_\infty = \pi^2 = 9.8696$ and 5.7831, respectively. Although the present method is not the best to use for uniform velocity profiles, we obtained the same Nu_∞ to five significant digits.

The Nusselt number for fully-developed temperature profile together with the square of the five first eigenvalues are reported in Table 1 for various values of α, or equivalently in terms of the power-law index n_c, both for parallel-plate channel and circular duct. The underlined values are in excellent agreement with those calculated through a pure numerical integration of the energy equation, (i.e. $Nu_\infty = 7.541$ for $m = 0$ and $\alpha = 2$ [8], $Nu_\infty = 3.567$ for $m = 1$ and $\alpha = 2$ [2, 6]), a semi-analytical procedure ($Nu_\infty = 3.95$ for $m = 1$ and $\alpha = 3$ [5]) and with the value $\pi^2/4 = 2.4674$ (slug flow, $m = 0$) obtained by using the classical procedure (A-solution).

Computation of the eigenvalues of higher order allows the determination of the temperature profile in any cross section of the duct and of the variations of the local Nusselt number from the inlet sections. Figures 1 and 2 show the variations of the bulk temperature and of the local Nusselt number as a function of Gz^{-1} in the thermal entry regions for both circular pipe and parallel-plate channel with constant surface temperature. The results are shown for Newtonian fluids ($n_c = 1$), for pseudoplastic fluids ($n_c = 0.5$ and 0.33) and for dilatant fluids ($n_c = 2$ and $n_c = \infty$). As it can be seen, the bulk temperature, the thermal entry length and the Nusselt number increase as n_c decreases. This is a well-known result. The asymptotic Nusselt numbers are shown in Fig. 2 for the ten cases considered in the present paper.

Values of the bulk temperature and local Nusselt number are reported in Table 2 for circular pipes and in Table 3 for flat ducts in the Newtonian case. It can be seen that these values are in excellent agreement with the results shown by Blackwell [2] and Ramachandran [5] for $X \geq 0.01$. It should be emphasized that the present results were obtained by using a maximum of 8 eigenvalues while the series expansions for the eigenfunctions had less than 200 terms. For smaller X-values, i.e. close to the duct inlet, convergence can be achieved but a larger number of eigenvalues is required. At higher truncatures of the solution expansion [equation (3)], it is not obvious to

Table 1. Nusselt number for fully-developed regime and square of the five first eigenvalues for Newtonian and power-law fluids: $^+$, dilatant fluid; $^{++}$, Newtonian fluid; and $^{+++}$, slug flow

α	n_c	Nu_∞	μ_0^2	μ_1^2	μ_2^2	μ_3^2	μ_4^2
1	∞^+	6.9532	3.47662	44.1385	129.258	258.795	432.746
2	1^{++}	7.5407	2.82776	32.1472	93.4749	186.805	312.136
3	0.5	7.9397	2.64658	28.1320	81.5671	162.900	272.125
4	0.33	8.2275	2.57109	26.1506	75.6684	151.030	252.232
5	0.25	8.4493	2.53317	24.9995	72.1724	143.958	240.359
6	0.20	8.6120	2.51183	24.2676	69.8724	139.278	232.483
7	0.166	8.7460	2.49886	23.7742	68.2533	135.961	226.887
∞	0^{+++}	9.8696	2.46740	22.2066	61.6850	120.903	199.859
			Flat ducts				
1	∞^+	3.2638	9.79169	61.3730	157.480	298.041	483.037
2	1^{++}	3.6568	7.31358	44.6095	113.921	215.241	348.564
3	0.5	3.9494	6.58236	39.0934	99.4962	187.795	303.987
4	0.33	4.1753	6.26298	36.3596	92.3260	174.141	281.798
5	0.25	4.3544	6.09623	34.7442	88.0602	165.994	268.545
6	0.20	4.4995	5.99940	33.6944	85.2458	160.598	259.749
7	0.166	4.6192	5.93896	32.9705	83.2592	156.769	253.496
∞	0^{+++}	5.7831	5.78318	30.4713	74.8870	139.040	222.932
			Circular pipes				

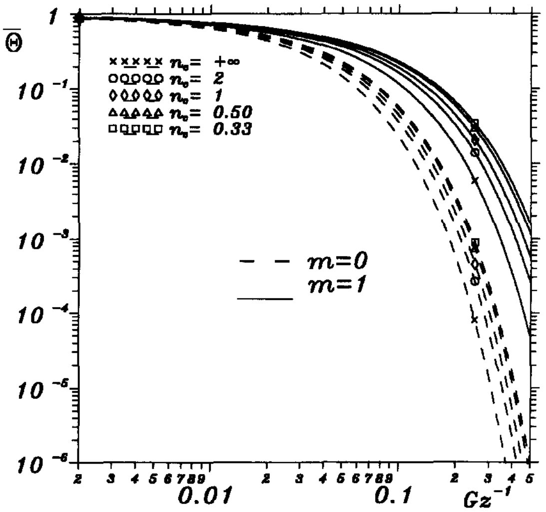

Fig. 1. Bulk temperature in the thermal entrance region of a circular duct and a parallel-plate channel for power-law fluids.

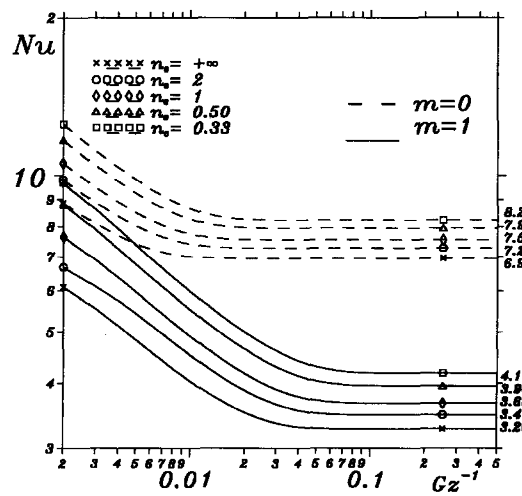

Fig. 2. Local Nusselt number in the thermal entrance region of a circular duct and a parallel-plate channel for power-law fluids.

Table 2. Bulk temperature and local Nusselt number for different X in the case of Newtonian fluid ($\alpha = 2$) in a circular pipe ($m = 1$)

		$\bar{\theta}(X)$			$Nu(X)$	
X	Ref. [2]	Ref. [5]	Present	Ref. [2]	Ref. [5]	Present
0.01	0.8362	0.8359	0.8362	6.002	5.961	5.990
0.02	0.7511	0.7511	0.7511	4.916	4.915	4.916
0.04	0.6280	0.6279	0.6280	4.172	4.177	4.172
0.10	0.3953	—	0.3953	3.710	—	3.710
0.20	0.1897	—	0.1897	3.658	—	3.658
0.40	0.0439	—	0.0439	3.657	—	3.657
1.00	0.0005	—	0.0005	3.657	—	3.657
2.00	0.0000	—	0.0000	3.657	—	3.657

Table 3. Bulk temperature and local Nusselt number for different X in the case of Newtonian fluid ($\alpha = 2$) in a flat duct ($m = 0$)

X	$\bar{\theta}(X)$	$Nu(X)$
0.01	0.9307	13.036
0.02	0.8908	10.729
0.04	0.8280	09.036
0.10	0.6883	07.784
0.20	0.5172	07.554
0.40	0.2937	07.541
1.00	0.0538	07.541
2.00	0.0032	07.541

accurately determine the eigenvalues with the numerical algorithm used. However, these eigenvalues differ by less than 1% from the asymptotic ones reported in Sellers *et al.* [1], i.e.

$$\mu_i = 16i/\sqrt{3}+20/(3\sqrt{3}) \quad \text{for } m = 0$$

$$\text{and } \mu_i = 4i+8/3 \quad \text{for } m = 1. \tag{15}$$

5. CONCLUSIONS

Based on series expansion of eigenfunctions on monomial functions of the positive real space, the present method for the Graetz problem in ducts with constant wall temperature has the main advantage of being easy to use. Indeed, the knowledge of the first eigenfunction yields highly accurate results for fully developed temperature profiles while few eigenvalues are enough to determine temperature distributions and local Nusselt number in the thermal entry region.

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